POINCARE'S THEOREM AND ITS USES

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ABSTRACT. The paper contains a general statement of the Virial Theorem which has been applied to find the periods of fundamental mode of oscillation of the electronic cloud of an atom and also of an atomic nucleus. The method adopted is the same as used by Ledoux (1945) and Pekeris (1941) for calculating the period of oscillation of a variable star. The method seems to give values of correct order for the above quantities.

INTRODUCTION

The present paper consists of the generalisation of well known Poincare's theorem $2T+\Omega=0$ for the steady state of a system of detached mass points or cloud of particles. The equation has been extended by Eddington (1926) to the form $\frac{1}{2}\frac{d^2I}{dt^2}=2T+\Omega$. In the above equations T is the kinetic energy of the particles, Ω their potential energy and I is the moment of inertia about some common origin. In his derivation of the above equation, Poincare assumes a potential between two particles of the form

$V_{ik}=-\frac{Gm_im_k}{r_{ik}}$ where V_{ik} is the interaction between two particles and r_{ik} is the mutual distance between them. The subscripts i and k denote the numbers of the particles. We shall now derive the same equation for a more general case.

1. *Derivation of Poincare's Theorem* for the case when the potential function is of the form $V_{ik}=f(r_{ik}^{-n})$, where n is an integer greater than one :

Consider the general motion of a cloud of particles and let the position of each particle be described by N generalised positional coordinates $q_1, q_2, q_3 \ldots\ldots q_r, \ldots\ldots q_N$, and let the corresponding force components acting on any particle be denoted by $X_1, X_2 \ldots X_r \ldots X_N$. Then according to Newton's laws of motion we have

$$m_k\frac{d^2q_r^k}{dt^2}=X_r^k,$$

where m_k is the mass of kth particle and q_r^k is rth positional coordinate of the particle whose equation of motion we have written and X_r^k, is the rth component of the force acting on this particle. We have now for kth particle

$$\frac{1}{2}\frac{d^2}{dt^2}\left(m_k q_r^{k^2}\right)=\frac{d}{dt}\left(m_k q_r^k \cdot \dot{q}_r^k\right)$$

$$=m_k \dot{q}_r^{k^2}+m_k q_r^k \ddot{q}_r^k$$

$$=m_k \dot{q}_r^{k^2}+q_r^k X_r^k. \qquad \dots \quad (1)$$

If we now consider all the particles we have

$$\frac{1}{2}\frac{d^2}{dt^2}\left(\sum_k \sum_{r=1}^{N} m_k q_r^{k^2}\right)=\sum_k \sum_{r=1}^{N} m_k \dot{q}_r^{k^2}+\sum_k \sum_{r=1}^{N} q_r^k X_r^k$$

$$\text{or} \quad \frac{1}{2}\frac{d^2 I}{dt^2}=2T+\sum_k \sum_{r=1}^{N} q_r^k X_r^k \qquad \dots \quad (2)$$

where I is again the moment of inertia of the system about the origin defined by $I=\sum_k \sum_{r=1}^{N} m_k q_r^{k^2}$ and T is the kinetic energy of the motion of particles forming the cloud. The second term on the right of equation (2) is the well known Virial of Clausius.

To evaluate the virial we consider two specific particles of masses m_i and m_k and at $(q_1^i, q_2^i, \dots q_r^i, \dots q_N^i)$ and $(q_1^k, q_2^k, \dots\dots q_N^k)$. Let the force exerted on the particle i due to the particle k have components A_1, A_2, ... A_N, then the force acting on the particle k due to particle i will have components $-A_1, -A_2, \dots -A_q \dots\dots -A_N$. Hence the contribution by these to the virial is

$$A_1(q_1^i-q_1^k)+A_2(q_2^i-q_2^k)+\dots\dots A_N(q_N^i-q_N^k), \qquad \dots \quad (3)$$

and if we consider all pairs of particles we have the total contribution by the whole cloud as

$$\frac{1}{2}\sum_i \sum_{k\neq i}\left[A_1(q_1^i-q_1^k)+\dots\dots A_N(q_N^i-q_N^k)\right]$$

$$=\frac{1}{2}\sum_i \sum_{k\neq i}\left[\sum_r A_r(q_r^i-q_r^k)\right] \qquad \dots \quad (4)$$

If $W_{ik}=f[r_{ik}^{-(n+1)}]$ is the force exerted between two particles we have various components of this force $A_1, A_2 \dots A_N$ etc. as

$$A_1=f(r_{ik}^{-(n+1)}) \times \frac{q_1^i-q_1^k}{r_{ik}}$$

and

$$A_r=f(r_{ik}^{-(n+1)}) \times \frac{q_r^i-q_r^k}{r_{ik}}$$

and hence the virial is

$$\frac{1}{2}\sum_{i}\sum_{k\neq i}\left[\sum_{r} f(r_{ik}^{-(n+1)}) \times \frac{q_r^i - q_r^k}{r_{ik}}(q_r^i - q_r^k)\right]$$

$$= \frac{1}{2}\sum_{i}\sum_{k\neq i}\left[\sum_{r} f(r_{ik}^{-(n+1)})\frac{(q_r^i - q_r^k)^2}{r_{ik}}\right]$$

$$= \frac{1}{2}\sum_{i}\sum_{k\neq i}\left(-\frac{\partial V_{ik}}{\partial r_{ik}} r_{ik}\right)$$

$$= \frac{1}{2} n \sum_{i}\sum_{k\neq i} V_{ik}$$

$$= n\Omega. \qquad \ldots (5)$$

Hence the equation (2) takes the form

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2T + n\Omega. \qquad \ldots (6)$$

2. If we take the potential function as well know Yukawa potential $V = \frac{A}{r}e^{-r/\lambda}$ so that we have $V_{ik} = \frac{A}{r_{ik}}e^{-r_{ik}/\lambda}$. In this case the virial would be

$$\frac{1}{2}\sum_{i}\sum_{i\neq k} -\frac{\partial V_{ik}}{\partial r_{ik}} r_{ik}$$

$$= \frac{1}{2}\sum_{i}\sum_{k\neq i} V_{ik} + \frac{1}{2}\times\frac{1}{\lambda}\sum_{i}\sum_{k\neq i} Ae^{-r_{ik}/\lambda}$$

$$= \Omega + \frac{1}{\lambda}\frac{\delta\Omega}{\delta(1/\lambda)}$$

$$= \Omega + \frac{1}{\lambda}\frac{\delta\Omega}{\delta\lambda}\cdot\frac{1}{\frac{\delta(1/\lambda)}{\delta\lambda}}$$

$$= \Omega - \lambda\frac{\delta\Omega}{\delta\lambda}. \qquad \ldots (7)$$

Therefore in this case Poincare's equation is

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2T + \Omega - \lambda\frac{\delta\Omega}{\delta\lambda}. \qquad \ldots (8)$$

3. *Application of Poincare's Theorem* for finding out the period of radial pulsation of Thomas-Fermi atom. For a could of electrical charge, various elements of which have an electrostatic interaction, we have the equation

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2T + \Omega,$$

where I is as before the moment of inertia with respect to the origin, T the kinetic energy and Ω is the electrostatic potential energy of the electron cloud of the atom. Fo a spherically symmetrical distribution of the charge, we have

$$I = \int_0^m r^2 dm(r)$$

where $m(r)$ denotes the mass interior to r and m is the total mass of the charge cloud. We shall consider the application of the above equation to the steady radial pulsation of a Thomas-Fermi cloud and in studying the problem we shall adopt Langrangian mode of description, in which we follow each particle (or element of mass) during its motion. Let the distance r form the centre of symmetry be used as such a Langrangian coordinate. Let δr denots the displacement from the equilibrium position r_0, The conservation of mass requires that

$$m(r_0 + \delta r) = m(r_0). \qquad \ldots (9)$$

If δI, $\delta\Omega$ and δT denote the corresponding changes from the equilibrium values in respective quantities at time t, we have

$$\tfrac{1}{2} \frac{d^2}{dt^2} (\delta I) = 2\delta T + \delta\Omega. \qquad \ldots (10)$$

Now to a first order in δr we have

$$\delta I = 2\int_0^m r\delta r\, dm(r) \simeq 2\int_0^m \frac{\delta r}{r_0} r_0^2 dm(r)$$

$$\simeq 2\int \frac{\delta r}{r_0} dI_0 \text{ where } I_0 = mr_0^2 = r_0^2 \int_0^m dm(r) \qquad \ldots (11)$$

The potential energy Ω of the charged cloud distributed unifomly in a sphere of radius r_0 surrounding a nucleus of charge Z is given by

$$\Omega_0 = -\frac{9}{10} \frac{Z^2 e^2}{r_0} \qquad \ldots (12)$$

So if the charged sphere assumes a radius r, the potential energy becomes

$$\Omega = -\frac{9}{10} \frac{Z^2 e^2}{r} \qquad \ldots (13)$$

and hence

$$\delta\Omega = +\frac{9}{10} \frac{Z^2 e^2}{r^2} \delta r$$

$$= -\frac{\delta r}{r_0} \times \frac{Z^2 e^2}{r_0} \times \frac{9}{10}$$

$$= +\Omega_0 \frac{\delta r}{r_0}. \qquad \ldots (14)$$

Since the charge cloud of the electron behaves as a degenerate electron gas and, therefore, the kinetic energy is given by

$$T = \tfrac{3}{5} \times \frac{h^2}{2m} \left(\frac{3n}{8\pi} \right)^{\frac{2}{3}} \times Z \qquad \ldots (15)$$

where Z is the number of electrons in the atom and n is the density of the electorn. The electron density n is

$$\frac{Z}{\frac{4\pi}{3} r_0^3}$$

and so we have

$$T_0 = \frac{3}{5} \times \frac{h^2 Z}{2 m r_0^2} \left(\frac{9Z}{32\pi^2} \right)^{\frac{2}{3}} \qquad \ldots \quad (16)$$

and in the displaced position r we have

$$T = \frac{3}{5} \times \frac{h^2 Z}{2 m r^2} \left(\frac{9Z}{32\pi^2} \right)^{\frac{2}{3}} \qquad \ldots \quad (17)$$

so

$$\delta T = -\frac{3}{5} \times \frac{h^2 Z}{m r^3} \left(\frac{9Z}{32\pi^2} \right)^{\frac{2}{3}} \delta r$$

$$= -\frac{2T_0}{r_0} \delta r. \qquad \ldots \quad (18)$$

Substituting these values of δI, $\delta\Omega$, δT in (10) we get

$$\frac{1}{2} \frac{d^2}{dt^2} \left(2 \int_0^m \frac{\delta r}{r_0} \delta I_0 \right) = -\frac{4T_0}{r_0} \delta r + \Omega_0 \frac{\delta r}{r_0} \qquad \ldots \quad (19)$$

Now for small periodic oscillations we can write

$$\frac{\delta r}{r_0} = \xi = \xi e^{i\sigma t}$$

In the evaluation of δT we may observe that T consists of two parts ; (1) the kinetic energy due to thermal motions and kinetic energy due to vibrations. It is evident that the latter is of second order in ξ and therefore can be ignored in a first order theory. Thus we have equation (19) as

$$\frac{d^2}{dt^2} \int_0^m \xi_0 e^{i\sigma t} dI_0 = -4T_0 \xi_0 e^{i\sigma t} + \Omega_0 \xi_0 e^{i\sigma t}$$

or

$$-\sigma^2 e^{i\sigma t} \int \xi_0 dI_0 = -4T_0 \xi_0 e^{i\sigma t} + \Omega_0 \xi_0 e^{i\sigma t}$$

or

$$\sigma^2 = \frac{4T_0\xi_0 - \Omega_0\xi_0}{\int \xi_0 dI_0}. \qquad \ldots \quad (20)$$

If we consider ξ_0 as constant we get

$$\sigma^2 = \frac{4T_0 - \Omega_0}{I_0}$$

For $Z = 1$, we have $\sigma^2 = \frac{6}{5}\left(\frac{h^2}{m r_0}\right)\left(\frac{9}{32\pi}\right)^{\frac{2}{3}} + \frac{9}{10}\frac{e^2}{r_0}$

$$\frac{3 \times (6.62)^2 \times 9^{2/3} \times 10^{33}}{(9.105)^2 \times (.528)^4 \times (4 \times 3.14 \times 3.14)^{2/3}} + \frac{9 \times (4.80)^2 \times 10^8}{9.105 \times (.528)^3}$$

$$= 91.73 \times 10^{32}$$

or $$\sigma = 9.58 \times 10^{16}.$$

From this we get the period of oscillation

$$\tau = \frac{2\pi}{\sigma} = \frac{2\pi}{9.58 \times 10^{16}}$$

$$= .104 \times 10^{-16} \text{ seconds}$$

4. *Oscillation period of a nucleus.*

Inside a nucleus composed of Z protons and N neutrons there are various types of interactions existing. We have at first the Yukawa potential of the form $A/r\, e^{-r/\lambda}$, then there is the Coulombian interaction between the protons. The Yukawa potential gives the neutron-proton interaction. In addition to these there are minute neutron-neutron interaction and exchange interactions. We shall here consider only first two which are important in the case of potential energy of the nucleus. We can write down the Poincare's equation for the nucleus as

$$\tfrac{1}{2}\frac{d^2 I}{dt^2} = 2T + \Omega - \lambda \frac{\delta\Omega}{\delta\lambda} + W \; ; \qquad \ldots \quad (22)$$

where Ω is the potential energy due to Yukawa interaction and W is the Coulomb interaction potential energy. Confining ourselves to the assumption that both the interactions are effective in the whole nucleus we get

$$W = \frac{3}{5} \times \frac{Ze^2}{R} \qquad \ldots \quad (23)$$

as before. The potential energy Ω due to the interaction of the form $\frac{A}{r} e^{-\frac{r}{\lambda}}$ can be found in a simple way.

Let P be a point outside the shell at a distance c from the centre O of a spherical shell. Consider a narrow zone of the shell QQ' at a distance r

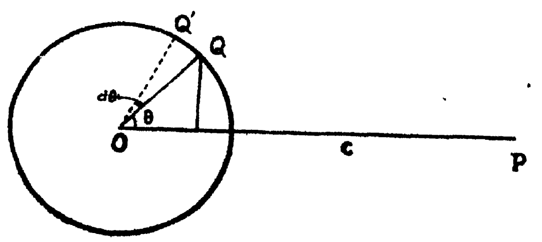

FIG. 1

from P. If a is the radius of the shell and θ denotes the angle QOP, the area of the zone is $2\pi a^2 \sin\theta\, d\theta$ so **that** if m is the mass per unit area of the zone, the potential at P due to this zone is

$$2\pi m a^2 \frac{A}{r} e^{-\frac{r}{\lambda}} \sin\theta d\theta \qquad \ldots \quad (24)$$

and so potential due to whole shell is

$$V = \int_0^{\pi} 2\pi a^2 m \sin\theta \frac{A}{r} e^{-\frac{r}{\lambda}} d\theta$$

$$= \frac{2\pi m a^2 A}{ac} \int_{c-a}^{c+a} e^{-\frac{r}{\lambda}} dr = \frac{4\pi m a A\lambda}{c} e^{-\frac{c}{\lambda}} \sinh\frac{a}{\lambda} \text{ for } c > a \qquad \ldots \quad (25)$$

and $\qquad V = \dfrac{2\pi m A a}{c} \displaystyle\int_{a-c}^{a+c} e^{-\frac{r}{\lambda}} dr$

$$= \frac{4\pi m A a}{c} \lambda e^{-\frac{a}{\lambda}} \sinh\frac{c}{\lambda} \text{ for } c < a \qquad \ldots \quad (26)$$

Using (25) and (26) we find the potential due to a solid sphere at a point whose distance is c from the centre of the sphere. If the point P is outside the sphere we have

$$V = \int_0^R 4\pi r^2 \times \frac{M}{\frac{4\pi R^3}{3}} \times \frac{A\lambda}{cr} e^{-\frac{c}{\lambda}} \sinh\frac{r}{\lambda} dr$$

$$= \frac{3MA\lambda e^{-\frac{c}{\lambda}}}{R^3 c} \left[R\lambda \cosh\frac{R}{\lambda} - \lambda^2 \sinh\frac{R}{\lambda} \right], \qquad \ldots \quad (27)$$

and if the point P is inside the sphere we have

$$V = \frac{4\pi}{c} \times \frac{3MA\lambda}{4\pi R^3} \left[\int_0^c r \sinh\frac{r}{\lambda} e^{-\frac{c}{\lambda}} dr + \int_c^R r e^{-\frac{r}{\lambda}} \sinh\frac{c}{\lambda} dr \right] \qquad \ldots \quad (28)$$

$$= \frac{3MA\lambda}{R^3 c} \left[\lambda c - (\lambda R + \lambda^2) e^{-\frac{R}{\lambda}} \sinh\frac{c}{\lambda} \right] \qquad \ldots \quad (29)$$

Using the equation (29) we can find Ω. It is given by
$\Omega = \frac{1}{2}\int V dm$

$$= \frac{1}{2} \int_0^R \frac{3MA\lambda}{R^3} \times \frac{1}{r} \left\{ \lambda r - (\lambda R + \lambda^2) \sinh\frac{r}{\lambda} \right\} \times 4\pi r^2 dr \times \frac{3M}{4\pi R^2}$$

$$= \frac{9M^2 A\lambda}{2R^3} \left[\lambda R^3 - (R\lambda + \lambda^2) e^{-\frac{R}{\lambda}} \left(R\cosh\frac{R}{\lambda} - \lambda^2 \sinh\frac{R}{\lambda} \right) \right] \qquad \ldots \quad (30)$$

Now from equation (22) we have

$$\frac{1}{2}\frac{d^2}{dt^2}(\delta I)=2\delta T+\delta\Omega-\lambda\frac{\delta}{\delta\lambda}(\delta\Omega)+\delta W \quad \ldots \quad (31)$$

and again

$$\delta I=2\int R\delta R dM(R)=2\int^{M}\frac{\delta R}{R}\,dI_0\;; \quad \ldots \quad (32)$$

$$\delta T=2T\frac{\delta R}{R}\;;\;\delta W=-W\frac{\delta R}{R} \quad \ldots \quad (33)$$

while

$$\delta\Omega=\frac{9M^2A}{4}\left[\left\{\frac{\lambda^2}{R}+\frac{3\lambda^3}{R^2}+\frac{4\lambda^4}{R^2}+\frac{\lambda^3}{R}+\frac{3\lambda^5}{R^3}\right\}+e^{-\frac{2R}{\lambda}}\left\{2\lambda+\frac{3\lambda^2}{R}\right.\right.$$

$$\left.\left.+\frac{\lambda^3}{R^2}-\frac{2\lambda^4}{R^2}-\frac{\lambda^3}{R}-\frac{3\lambda^5}{R^3}\right\}\right]\frac{\delta R}{R} \quad \ldots \quad (34)$$

and

$$\lambda\frac{\delta\Omega}{\delta\lambda}=\frac{9M^2A}{4}\left[2\lambda+\frac{27\lambda^3}{R^2}+\frac{64\lambda^4}{R^2}+\frac{9\lambda^3}{R}+\frac{75\lambda^3}{R^1}\right]e^{-\frac{2R}{\lambda}}\frac{\delta R}{R}$$

$$+\frac{9M^2A}{4}\left[\frac{2\lambda^2}{R}+\frac{9\lambda^3}{R^2}+\frac{16\lambda^4}{R^2}+\frac{3\lambda^3}{R}+\frac{15\lambda^5}{R^3}\right]\frac{\delta R}{R}$$

$$-\frac{9M^2A}{2}\left[2+\frac{3\lambda}{R}+\frac{\lambda^2}{R^2}-\frac{2\lambda^3}{R^2}+\frac{\lambda^2}{R}-\frac{3\lambda^4}{R^3}\right]e^{-\frac{2R}{\lambda}}\frac{\delta R}{R} \quad \ldots \quad (35)$$

Putting $\frac{\delta R}{R}=\xi=\xi_0 e^{i\sigma t}$ and substituting these in (31) we get as before

$$\sigma^2=\frac{4T+W}{MR^2}-\frac{9M^2A}{4R^2}\left[\frac{3\lambda^2}{R}+\frac{12\lambda^3}{R^2}+\frac{20\lambda^4}{R^2}+\frac{4\lambda^3}{R}+\frac{18\lambda^5}{R^3}\right]$$

$$-\frac{9MA^2}{4R^2}e^{-\frac{2R}{\lambda}}\left[4\lambda+\frac{9\lambda^3}{R}+\frac{4\lambda^2}{R^2}-\frac{10\lambda^4}{R^2}+4\lambda^3-\frac{18\lambda^5}{R^4}\right]$$

$$+\frac{9M^2A}{2R^2}e^{-\frac{2R}{\lambda}}\left[2+\frac{3\lambda}{R}+\frac{\lambda^2}{R^2}-\frac{2\lambda^3}{R^2}+\frac{\lambda^2}{R}-\frac{3\lambda^4}{R^3}\right] \quad \ldots \quad (36)$$

Now for a nucleus with Z protons and N neutrons the kinetic energy of the nuclear particles is

$$T=\frac{3}{5}\times\frac{h^2}{4\pi^2M_p^2}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N^{5/3}+Z^{5/3}}{R^2}$$

$=2980$ M. e. volts for a nucleus with $Z=80$ and $N=120$ *i.e.* Hg^{200}.

For the same $\lambda = 2.37 \times 10^{-13}$ cm and $R = 8 \times 19^{-13}$ cm. The mass of the nucleus is 200 A. M. units. Substituting these we obtain

$$T_{\text{nucleus}} \simeq 10^{-21} \text{ seconds.}$$

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